

History and epistemology in mathematics education

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The tradition of setting mathematics education in a historical or epistemological perspective is well rooted in Italy. The historian Gino Loria was a pioneer in advocating the use of history in mathematics teaching, especially in relation with teacher training, see (Loria, 1899). He followed the idea - widespread in those years in the mathematical community - that mathematics teachers need to revisit elementary mathematics from an advanced stand point¹. Many authors (Heppel, 1893; Klein 1896) indicated history of mathematics as an efficient means to perform this revisiting. In those years (at the turn of the century) there was also a broad discussion on possible uses of history in the classroom. Under the influence of the recapitulation law - i.e. ontogeny recapitulates phylogeny - stated by the German biologist Ernst Haeckel (1874) it was advocated that teaching sequences in mathematics should follow human development in mathematics. This theory had its apotheosis in the famous book by Benchara Branford (1921). This work was known in Italy: in the library that is Loria's bequest to the University of Genoa there is a copy of the 1921 edition with the author's dedication to Loria. Nevertheless neither Loria nor other Italian authors make explicit reference to the recapitulation law when they advocate resort to history in mathematics teaching.

Starting from the turn of the century the Italian school environment was strongly influenced by the movement of rigor and the discussion on the foundations of mathematics, as well as by the rising of logic as a branch of mathematics. We have to remember that in that period Peano and his scholars were active researchers in Italy. In those years national programs and several new textbooks were influenced by this new fashion and interesting discussions about pros and cons of a strictly rigorous approach are present in mathematics journals for teachers published in that period. A good balance of different orientations was reached by Federigo Enriques. He was not only a great mathematician leading the Italian geometric school, but also a leader in the discussion on mathematics education. He edited an important book for teachers *Questioni riguardanti la geometria elementare* (Zanichelli, Bologna, 1900), (soon translated into German by initiative of Klein). With the title *Questioni riguardanti le matematiche elementari* this work had many editions (first 1912, second 1914, third 1924-27) and contributed to developing the mathematical culture of generations of teachers. In his approach to mathematics education problems, Enriques stressed the importance of epistemology and history. This approach, together with his interest in problems of knowledge, is already evidenced in the talk delivered in the first important international meeting of the new born ICMI held in Milan, (Enriques, 1911). The

¹ An interest for history can still be traced in the national official programs: the historical approach is explicitly mentioned in the premises to the list of mathematical contents (Recommendations for teachers). Moreover in many in-service teacher training courses organized by the Italian Ministry of Education (see section 3 of the first part of this volume) history is contemplated as part of the program.

historical perspective of mathematics teaching held by Enriques is combined with a view strongly oriented toward epistemology, see (Enriques, 1921). This orientation characterizes part of the Italian studies in mathematics education, as shown in (Speranza & Grugnetti, 1996). In particular, the contribution of Francesco Speranza in the second half of 20th century has been very important in Italy as for the relationships among epistemology, history and didactics, see (Malara et al., 2000; Marchini, 2001).

In this chapter we report on works in the field in question carried out by Italian authors in the years from 1999 to 2003. Our overview is not exhaustive, but we hope that it gives a faithful idea of the current state-of-the art in Italy. Some of the authors are school teachers and their works highlight the actual implementation of some theoretical ideas emerging from the papers authored by university researchers in education and in history. In our analysis we distinguish between works dealing with history and pedagogy and works with a more marked epistemological orientation, although the boundary between the two domains is fuzzy, as we will see in the following.

Broadly speaking the papers dealing with history fall in the following (sometimes overlapping) categories:

- (1) History of mathematical concepts
- (2) History in teacher training
- (3) History in the classroom

In categories (1) and (2) we find papers such as (Chimetto & Zoccante, 2002 and 2003; Ferrari, 2002, Ferrari & Pizzetti, 1999 and 2001; Grugnetti, 2000) that deal with the historical development of concepts that intervene in school mathematics (algorithms, cartography, rule of signs, angles, angles of contingency). A particular aspect concerns the history of mathematics as a tool for investigating the processes of teaching/learning in multicultural situations (Spagnolo, 2002 and 2003): through a comparison of the history of deducing and showing in the Chinese and the European culture, different schemes of reasoning are studied.

The aim of the papers (1) is twofold. On one hand the development of mathematical ideas helps the teacher overcome epistemological obstacles. For example, in (Bagni, 2000b) a comparison of historical definitions of similar figures proposed in the classroom brings pupils (aged 12-13 years) to consider different approaches and to compare their consequences in constructing mathematical objects. On the other hand the papers in question answer to teachers' need of knowing history in an organized way. They not only give information, but also provide materials for reflecting on teaching/learning difficulties intrinsically inherent to certain mathematical concepts.

Some papers of category (2) explicitly outline a teaching sequence for courses aimed at training or re-training teachers. To make this sequence really effective it is necessary to consider firstly learning difficulties and to select historical materials as a consequence of this choice. For example, if we consider algebra, firstly we single out the following critical points on this subject:

- symbolism
- the relation between arithmetic and algebra
- the relation between geometry and algebra
- giving meaning to manipulation
- the obstacle of formalism (generalization, abstraction,...)

Afterwards we go along the history of algebra to discover moments where the previous points appear, as shown in (Sfard, 1995). In this way teachers become aware of the

difficulties encountered by students and have the means to study the nature of these difficulties. It is clear that the epistemological nature of obstacles plays a relevant role in this process. This method of working in parallel in mathematics education and history was applied in (Furinghetti & Somaglia, 2001) taking as a common thread the method of analysis in the history of algebra.

In order to train teachers and to plan teaching sequences it may be useful to consider how the teaching of a certain topic has developed. Becchere and Tazzioli (2000) have used this method to discuss the concept of volume by going along the different presentation of this concept in the textbooks from 1861 (the birth of Italy as a National state) to nowadays. Pergola, Zanolì, Martínez and Turrini, (2001; 2002a and 2002b), Pergola and Maschietto (2003) have studied mathematical models and mechanical curve-tracing tools. Bartolini Bussi (2000 and 2001) analyses the didactical use of all these artifacts in the classroom. The interest for these concrete tools is emerging and disappearing every now and then, see (Togliatti, 1922). Mammana and Tazzioli (1999) analyze how the development of mathematical research has influenced aspects of mathematical instruction.

We find particularly interesting the papers in category (3), since they can show us how theoretical ideas are put into practice. As a matter of fact the problem of using history in mathematics teaching is widely discussed from different points of view. There are famous mathematicians who advocate this use: some of them actually applied their ideas, see (Weil, 1980 and 1984); other authors discuss pros and cons, see (Bagni, 2000a; Menghini, 2000; Siu, to appear). Among mathematics educators Freudenthal (1973) is promoting the use of history through “guided reinvention”. Barbin (1994) introduces the concept of *depaysement* (alienation): she explains that integrating history in mathematics replaces the usual with something different and challenges one’s perceptions through making the familiar unfamiliar. As it happens when a person is set in a foreign landscape, after an initial phase of bewilderment there are attempts of replacement and orientation. This phenomenon is particularly suitable to be used in courses for teacher education (pre-service or in-service). Through *depaysement* and the following phases of replacement and orientation, history offers to teachers the opportunity to rethink their ideas on the nature of mathematical concepts and on the processes of construction of these concepts on behalf of students, see (Furinghetti, 2000a). The papers in category (3), i.e. the papers about history in the classroom, allow checking through experiments the actual facilities offered by the use of history. In the same time, they show the limitations and directions in using history in the classroom.

The fact that the history of mathematics is integrated in mathematics teaching emerges from various Italian papers. This means that, as we have explained in the case of teacher training, the choice of historical topics is made in connection with a careful analysis of students’ difficulties. In this way, history is not a further burden in a crowded curriculum, rather it mediates teaching. Most of the authors are secondary school teachers and we feel that they are using history because they are convinced of its validity as mediator of mathematical knowledge. The papers (Bagni, 2001a and 2001b; Furinghetti & Paola, 2003; Paola, 2000; Rottoli & Riva, 2000) describe experiments carried out with students aged 15 onwards, while (Demattè, 2003; Foschi, 2003; Gregori, 2001) deal with topics suitable also for younger students.

Among the experiences carried out in the classroom we mark out the experiment carried out by Testa (2000), who attempts to introduce students to the methodology of

historical investigation. We have found also examples of dramatization, see (Prosperini, 1999).

We know that one of the problems inherent in history is the use of original sources. Some experiments are based on original sources: Lagrange and Pacioli by Paola, old texts of arithmetic by Demattè and Gregori... This means that in the suitable conditions it is possible to act according to the suggestions of researchers in mathematics education, see chapter 9 in (Fauvel & van Maanen, 2000) and (Arcavi, & Bruckheimer, 2000).

Commenting on their experiments sometimes the teachers point out that their students' construction of mathematical objects follows the same path of the historical development and thus their trust in the efficacy of the use of history relies on this fact. From the description of the various experiments and from the teachers' comments we feel that usually teacher uses history to guide their students in the construction of mathematical objects not by following exactly the paths of the ancestors, but by discussing and confronting modern and old views (see Freudenthal guided reinvention).

As for the epistemological side of research in mathematics education we note that several works are primarily focused on epistemological representations (i.e. representations of the possible cognitive ways referred to some particular mathematical concepts) and on historical-epistemological representations. In this perspective the various and complex interactions between history and didactics require important epistemological assumptions (Radford, 1997): for instance, selection of historical data is relevant, and moreover several problems are connected with their interpretation, always based upon cultural institutions and beliefs.

Previously we have pointed out that to make efficient the use of history in mathematics teaching one has to integrate history in the curriculum and not to add it. This means that it has to be put in relation with theoretical paradigms, such as epistemological obstacles, the socio-cultural approach, the "voices and echoes" perspective, experimentally found historical conceptions. In this concern we mention the papers (D'Amore, 2000), (D'Amore & Fandiño Pinilla, 2002), (Bagni, 2002), (Furinghetti & Somaglia, 2003), (Grugnetti et al. 2000), (Radford, Boero & Vasco, 2000), (Scimone, 2003), (Speranza & Vighi, 2002). The study of epistemological obstacles has been undertaken through a theoretical-experimental modelling (Spagnolo, 1999 and 2002) and the transition from arithmetic to algebra is studied in (Di Leonardo, Marino & Spagnolo, 2000).

In the mentioned papers the interlacement between history and epistemology is rather evident. Epistemology may also act as a support for the construction of curricula, since it entails some interesting reflections concerning the reorganization of the curriculum in a perspective of educational innovation, see, for example, (Pellegrino, 2001). The influence of different epistemological choices upon curriculum development is also examined by Fandiño Pinilla (2002).

Epistemology may also be an efficient support for teacher training courses, since it fosters a meaningful reorganization of knowledge (*Savoir*) in the process of teaching-learning. In this concern we quote the papers by Pellegrino and Fiori dealing with the bases of geometry (2000a) and with the bases of arithmetic (2000b and 2001). Another very important aspect regards epistemological reflection related to both concepts and objects of mathematics (D'Amore, 2000b). Some important epistemological choices emerge both in the work of teachers, and in the direction taken by society when orienting teachers training (Fandiño Pinilla, 2001); moreover, some different meanings and the relevance of

competence are discussed from a variety of epistemological perspectives (Fandiño Pinilla, 2003).

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Works on history and epistemology in mathematics education published in the years 1999-2003 by Italian authors

Bagni, G.T.: 2000a, 'The role of the history of mathematics in mathematics education: reflections and examples', in I. Schwank (editor), *Proceedings of CERME-1* (Osnabrück), v.II, 220-231.

The effectiveness of the use of the History of Mathematics in Mathematics Education is worthy of careful research: for instance, the introduction of some fundamental algebraic

concepts can be based upon a historical approach. We considered the introduction of the group concept to an experimental sample of High School students aged 16-18 years through a historical example drawn from Bombelli's *Algebra* (1572); a second sample of students was given a parallel introduction through a Cayley table (both samples were asked the same test questions). We conclude that a pre-axiomatic start to group theory can be useful, but it is not always enough to guarantee full learning.

Bagni, G.T.: 2000b, 'Matematica e bellezza, bellezza della matematica', *Rivista di Matematica dell'Università di Parma*, v.6, n.3, 51-61.

In this paper we present three historical examples in order to discuss some relations connecting mathematics and Beauty. An epistemologically relevant example of beauty in Mathematics can be pointed out in the comparison of three proofs by Euclid, by Leonhard Euler and by Paul Erdős of the celebrated theorem which states that the set of prime numbers is infinite. Such proofs can be usefully considered in teachers training courses.

Bagni, G.T.: 2001a, 'Infinito e infinitesimo potenziale ed attuale: una sfida per la Scuola Secondaria Superiore', *Bollettino dei Docenti di Matematica*, n.42, 9-20.

Different ideas and expressions of infinitesimal methods in history and in mathematics education are investigated, with particular reference to the limit notion in high school. The influence of semiotic registers upon didactics is examined and its effectiveness for the full characterization of concepts is evaluated: we noticed that historical development of representation registers can be compared with the parallel development of the notion in students' minds. However the study of the problems encountered in the historical development of a concept does not necessarily help students with their difficulties, because of the different knowledge structures and beliefs: so the educational use of examples from History must be carefully controlled in order to obtain full learning.

Bagni, G.T.: 2001b, 'La introducción de la historia de las matemáticas en la enseñanza de los números complejos. Una investigación experimental en la educación media superior', *Revista Latinoamericana de Investigación en Matemática Educativa*, n.1, 45-62.

In this paper the idea of imaginary numbers in the learning of Mathematics is investigated, referred to Italian High School (*Liceo scientifico*, pupils aged 16-18 years): after a historical preface, the status of this concept is studied through some tests and interviews. We point out that several pupils accept the presence of imaginary numbers when they appear in the resolution of an equation, but they refuse them if they are the final result of an equation. We conclude that the effectiveness of historical examples in classroom practice must be controlled by teachers through experimental methods.

Bagni, G.T.: 2002, 'Mathematical formal models for the learning of Physics: the role of an historical example', in M. Michelini & M. Cobal (editors), *Developing formal thinking in physics. First international Girep seminar 2001. Selected papers*, Forum, Udine, 188-191.

History of Sciences is an important tool for Didactics: in this paper we propose the introduction of the concepts of work and of kinetic energy through a geometric example

based upon Gian Maria Ciassi's work (1677). Sometimes historical development of a concept is not suitable in order to plan curricula; however frequently there is an analogy between the stages of the historical development and corresponding educational stages. Of course, processes of teaching-learning take place nowadays: so educational work can be based upon the results achieved in the full historical development and History makes it possible to point out mathematical formal models that can be used in Didactics of Physics by analogy. This correspondence is an important tool for teachers: of course a deep epistemological skill is needed and this is a matter related to teacher training.

Bartolini Bussi, M.G.: 2000, 'Ancient instruments in the mathematics classroom', in *ICMI Study*, 343-351.

The history of mathematics can enter classroom activity by investigating copies of ancient instruments and other artefacts, reconstructed on the basis of historical sources. Several examples are discussed concerning arithmetic, geometry and application of mathematics. Possible ways of introducing them into the classroom activity are discussed. A special emphasis is given to the collection of geometrical instruments reconstructed in Modena.

www.macchinematematiche.unimo.it > theatrum machinarum

www.macchinematematiche.unimo.it > perspectiva artificialis

Bartolini Bussi, M.G.: 2001, 'The geometry of drawing instruments: arguments for a didactical use of real and virtual copies', *Cubo Matematica Educacional*, v.3, n.2, 27-54.

This paper starts with a play in three acts with a prologue and a provisional epilogue. The protagonists are mathematics entities, i.e. (abstract) curves and (theoretical) instruments (drawing instruments realised by linkworks), but their voices are uttered and commented by human characters. The parts are set in different ages. In the prologue, the voice is Euclid's one, as a representative of the geometers of the classical age; in the act one, the voices are uttered by geometers of the 17th and 18th centuries; in the act two, Kempe recites, as a representative of a group of French and British amateur linkworkers of the 19th century; the act three is set nowadays and Thurston is evoked on the scene. Then the potentialities of the introduction of the field of experience of drawing instruments in the mathematics classroom are discussed, with an a priori comparison between the activity with material linkworks and the activity with virtual 'copies'.

Bartolini Bussi, M.G. & Sierpinska, A. et al.: 2000, 'The relevance of the historical studies in designing and analyzing classroom activities', in *ICMI Study*, 154-161.

When a mathematics educator draws on the history in designing activities for the students he or she may be looking for geneses of mathematical ideas or contexts of emergence of mathematical thinking. The aim is defining conditions which have to be satisfied in order for the students to develop these ideas and this thinking in their own minds. In this paper some examples presented in the Luminy Study Meeting are discussed. They concern several fields among which geometry, linear algebra and calculus.

Becchere, M.M. & Tazzioli, R.: 2000, 'Il concetto di volume nei libri di testo: una analisi storico critica nell'ambito dei programmi (1867–1986)', *L'Educazione Matematica*, a.XXI, s.VI, v.2, 102-115.

This paper looks into the idea-concept of volume mastered by students from 6 to 24 years old, teachers and other categories of adults. An initial research, carried out through a chart submitted to all categories, has pointed out that wrong convictions about the concept of volume tend to persist (for example, solids that are not similar to the stereotypes studied at school like the pipe, have been often showed as without a volume). Starting from these results, investigation has been oriented to give answers to the questions: what is the school influence on the formation of the concept of volume and, in particular, are text-books liable for the birth of erroneous opinions a lot of people have? Definitions of volume adopted in the treatises (hand-books) beginning from the Coppino act (1867) up to the Falcucci bill (1986) has been compared with methodologies suggested also in an implicit way. A historic-critical study has pointed out that the idea of volume often conflicts with the Euclidean definition of space taken up by the object.

Chimetto, M.A. & Zoccante, S.: 2002, 'Antichi algoritmi e nuove tecnologie', *Progetto Alice*, v.III, 63-78.

Tools that new technologies offer are especially suitable to revisit old algorithms present in the history of mathematics. Although these algorithms are no longer used, they fit to bring up new topics in mathematics education. In this work, we introduce logarithms using an algorithm, already described by Euler in *Introductio in Analysin infinitorum*, implemented and analyzed through symbolic calculators.

Chimetto, M.A. & Zoccante, S.: 2003, 'Geometria e geografia I problemi della cartografia: progetto per la produzione di materiali', *L'Insegnamento della Matematica e delle Scienze Integrate*, v.26A-B, 749-760.

The aim of this paper is to discuss the introduction of elements of cartography in the classroom. Although the use of history is not a declared aim of the paper, history is always present in the paper (and in the described related activities). The maps of Archimedes, Mercator, Lambert, Gauss, Gall are analyzed.

D'Amore, B.: 2000a, 'La didáctica de la matemática a la vuelta del milenio: raíces, vínculos e intereses', *Educación Matemática*, v.12, n.1, 39-50.

This article endeavours to identify some future developments in mathematics research, which of the current studies will continue and what factors might favour the decline of some areas of research considered important today. The author focuses on how some of the classical areas of research into Mathematics Education (such as "contract", "situation", "obstacle", etc.) still offer elements of great interest and require further research. Moreover, an examination of the relationship between noesis and semiotics and these areas is proposed.

D'Amore, B.: 2000b, “‘Concetti” e “oggetti” in matematica’, *Rivista di Matematica dell’Università di Parma*, v.6, n.3, 143-151.

In this article various interpretations of the terms ‘concept’ and ‘object’ in Mathematics are analyzed, using the History of Philosophical Thought, Psychology, and the recent “anthropological” perspective, demonstrating how it could be necessary to enter a “pragmatic” theory. The most interesting aspect of these studies is that they link the classical research of Brousseau to more recent developments in the debate on the nature of concepts, their acquisition and semiotic representation. The research has a strong epistemological bias.

Related papers

D’Amore B.: 2001, ‘Un contributo al dibattito su concetti e oggetti matematici: la posizione “ingenua” in una teoria “realista” vs il modello “antropologico” in una teoria “pragmatica”’, *La Matematica e la sua Didattica*, n.1, 4-30. [Spanish transl.: ‘Una contribución al debate sobre conceptos y objetos matemáticos’, *Uno*, v.27, 2001, 51-76. French transl.: ‘Une contribution au débat sur les concepts et les objets mathématiques’, *Scientia Paedagogica Experimentalis*, XXXVIII, v.1, 2001, 17-46].

D’Amore, B. & Fandiño Pinilla, M.I.: 2001, ‘Concepts et objets mathématiques’, in A. Gagatsis (editor), *Proceedings of the third intensive programme Socrates-Erasmus* (Nicosia), 111-130.

D’Amore, B.: 2003, ‘The noetic in mathematics’, *Scientia Paedagogica Experimentalis*, v.XXXIX, n.1, 75-82.

D’Amore B. & Fandiño Pinilla, M.I.: 2002, ‘Un acercamiento analítico al “triángulo de la didáctica”’, *Educación Matemática*, v.14, n.1, 48-61.

In this work we take up again some considerations derived from the literature, suggesting an analytical lesson from the triangle student-teacher-knowledge intended as a systematic model of fundamental didactics. The idea is not to get involved in the intrinsic complexity of the model, but only to suggest a way of articulating the main reflections which various authors have made on this subject. This analysis is based not only on methodologies and previous studies deriving from the work of Brousseau, but also on choices with a strong epistemological bias.

Demattè, A.: 2003, ‘Aritmetica in classe con «Larte de labbacho»’, *Scuola e Didattica*, a.XLIX, n.6, 39-42.

The author of this paper is a secondary school teacher who relies on the history of mathematics as an artifact to use in classroom as a mediator of mathematical knowledge. He assumes that the use of original sources is the most efficient way to use history. In this paper he outlines some teaching sequences based on the book *Larte de labbacho*, printed in Treviso (Northern Italy) (December 10th, 1478). On one hand the book allows to develop problems of proportionality, properties of numbers and operations on them. On the other hand the teacher may introduce students to the elaboration on hypotheses on the origin of this anonymous manual.

Di Leonardo, M.V., Marino, T. & Spagnolo, F.: 2000, ‘The “0”, is it an obstacle?’, in A. Rogerson (editor), *Proceeding of international conference on mathematics education into*

the 21st Century (Amman), 289-297. (In Italian: 1999, 'Lo "0" è un ostacolo?', *Quaderni di Ricerca in Didattica*, n.8, 91-109.)

<http://dipmat.math.unipa.it/~grim/21project.htm>

To decide whether "zero" is an epistemological obstacle we consider the theoretical/experimental model of Brousseau-Spagnolo. In this paper we analyze the epistemological representations of some historical traces of "zero". We also present the related experimental study.

Fandiño Pinilla, M.I.: 2001, 'La formazione degli insegnanti di matematica: una cornice teorica di riferimento', *La Matematica e la sua Didattica*, n.4, 352-373.

This paper is a short theoretical treatise on the subject: mathematical teacher training, with some considerations based on the recent history of this topic. As well as presenting a critical overview of the field, the author defines various situations and choices with clear epistemological bases, for example, by presenting various possible developments for work and research according to the initial epistemological position assumed, which influences and characterizes directions taken. The epistemological choice not only emerges in the work of teachers, but also in the direction taken by Society when orienting their training.

Related paper

D'Amore, B. & Fandiño Pinilla, M.I.: 2003, 'La formazione iniziale degli insegnanti di matematica', *La Matematica e la sua Didattica*, n.4, 413-440.

Fandiño Pinilla, M.I.: 2002, 'Ipotesi alla base di un curriculum di matematica', *La Matematica e la sua Didattica*, n.4, 369-410.

This paper considers some examples, taken from recent international literature, which offer a basis for the construction of a Mathematics curriculum. Particular emphasis is given to epistemological and didactic questions, with proposals for interpretation and analysis based on the author's international experience in the field of teacher development. Above all, emphasis placed on epistemological aspects is developed on the basis of a pragmatism juxtaposed to realism. Thus the way in which curriculum development is influenced more by epistemological choices than by those related to content is demonstrated.

Fandiño Pinilla, M.I.: 2003, "Diventare competente", una sfida con radici antropologiche' *La Matematica e la sua Didattica*, n.3, 260-280.

In this paper we propose a pragmatic-anthropological foundation for mathematics education "for competencies", providing a perspective based on didactic practice, also in relation to teacher training. The significance of competence is discussed from a variety of epistemological perspectives, with emphasis placed on the strong relationship between Chevallard's "anthropological ideas" and the pragmatic choices that characterize them and, in the author's opinion, prepare these ideas on an epistemological ground. The work is aimed at both teacher's practice and didactic research into methodological principles, neither of which can ever be independent of basic epistemological choices.

Ferrari, M.: 2002, 'La regola dei segni e la sua storia', *L'Insegnamento della Matematica e delle Scienze Integrate*, v.26B, 313-345.

The article describes the history of the rule of signs for the multiplication of integer numbers from Diophantus to Hankel. It also studies the different types of justification given by the authors, who wrote for "beginners".

Ferrari, M. & Pizzetti, F.: 1999, 'Il concetto di angolo nella matematica greca', *L'Insegnamento della Matematica e delle Scienze Integrate*, v.22B, 407-431.

Greek genius has given rise to a number of remarkable ideas and discoveries, one of which, the concept of angle, developed by illustrious Greek philosophers and mathematicians and related to Aristotle's theory of Categories, is particularly interesting. The various notions of angle provoked continuous debate until the 18th century, the issue being whether the angle should be considered as belonging to the category of quality (Aristotle), quantity (Apollonius, Carpus, Plutarcus) or relation (Euclid).

In the 5th century, the neoplatonic Proclus tried to settle the controversy by proposing a synthetic vision of the angle but even he was unable to solve the problem of reconciling the quantitative aspect of angle of contingency (a horn-like angle) with the fact that the entity of the plane angles is non-Archimedean by Prop. III, 16 of Euclid's 'Elements'.

Ferrari, M. & Pizzetti, F.: 2001, 'Gli angoli di contingenza nell'antica Grecia', *L'Insegnamento della Matematica e delle Scienze Integrate*, v.24B, 211-233.

In the debate on the nature of the angle – qualitative, quantitative or relational- there come to develop from the thirteenth century onwards many an issue on the character of the angle of contingency (delimited at one point by a circumference and its tangent). In ancient times already many accurate investigations were carried out probably by Democritus of Abdera and definitely by Euclid and Apollonius of Perga on this geometric object: It is the aim of this paper to try and reconstruct them.

Ferrari, P.L.: 2003, 'Abstraction in mathematics', in L. Saitta (editor), *The abstraction paths: from experience to concept*, *Phil.Trans.R.Soc.Lond.* B, v.358, n.1435, 1225-1230.

Some current interpretations of abstraction in mathematical settings are examined from different perspectives, including history and learning. It is argued that abstraction is a complex concept, and that it cannot be reduced to generalization or decontextualization only. In particular, the links between abstraction processes and the emergence of new objects are shown. The role that representations play in abstraction is discussed, taking into account both the historical and the educational perspectives. As languages play a major role in mathematics, some ideas from functional linguistics are applied in order to explain to which extent mathematical notations are to be considered abstract. Finally, abstraction is examined from the perspective of mathematics education, to show that also the teaching ideas resulting from one-dimensional interpretations of abstraction have proved utterly unsuccessful.

Foschi, A.: 2003, 'L'uso delle fonti storiche nelle lezioni di matematica: misurare un'altezza della quale sia inaccessibile la cima', *L'Insegnamento della Matematica e delle Scienze Integrate*, v.26B, 149-166.

In this article we offer resource material for lessons about theorem 19 in Euclid's "Optics" and some of its related topics, such as: properties of similar right triangles and the propagation and reflection of light. The central part of this paper originates from real classroom lessons. Our aim is presenting one of our teaching experiments to increase pupils' interest in mathematics and physics. In particular, using the history of these two subjects, we have shown how the interrelations between physical ideas and mathematical theoretical arguments have been revealed to our students. The role of history as an integral part of mathematics education and as a motivating force for both practical and theoretical geometry is briefly discussed too. Our aim is to contribute from a useful point of view to the published essays which contain suggestions for resources or lessons.

Furinghetti, F.: 2000a, 'The history of mathematics as a coupling link between secondary and university teaching', *International Journal of Mathematical Education in Science and Technology*, v.31, 43-51.

Often mathematics students develop a very poor conception of mathematics and its teaching during the years they spend in university. This fact is bad in all cases, but even more in the case of those students who will be mathematics teachers in school. In this paper I argue that the history of mathematics may be an efficient element to provide students with flexibility and motivations towards mathematics. The theoretical background of my work draws both on recent research in mathematics education and on papers written by mathematicians of the past. I support my opinion with examples. One example concerns a historical presentation of 'definition'; it was developed with mathematics students who will become mathematics teachers. As for the students oriented to research or to applied mathematics, I present an example to address the problem of the secondary-tertiary transition.

Furinghetti, F.: 2000b, 'The long tradition of history in mathematics teaching: an old Italian case', in V. Katz (editor), *Using history to teach mathematics: An international perspective*, Mathematical Association of America, 49-58.

The use of history in mathematics teaching is a matter of discussion very much alive both in the world of historians and of mathematics educators. It may be encouraging for researchers and teachers involved in this kind of study to know that there is a long tradition behind the experiences and the discussions carried out at present; evidence of this fact is offered by the old mathematical journal for secondary school students which is presented in this paper. The interest of this journal in connection with the debate on the use of history in mathematics teaching mainly relies on two facts: The journal is one of the few (the only?) journals of its time which had the declared aim of introducing the history of mathematics in classroom practice. The great majority of the authors who published articles in it were teachers and thus their suggestions really came from a practical knowledge of classroom life. The analysis of the journal gives the occasion to discuss some issues related to the subject "history in the classroom".

Furinghetti, F.: 2003, 'Storia della matematica per insegnanti e studenti', in E. Castro, P. Flores, T. Ortega, L. Rico & A. Vallecillos (editors), *Proceedings VII Simposio SEIEM* (Granada), 87-95.

We consider the following questions: Is it necessary to know the history of mathematics? How much history do we have to know? How do we have to know the history? The answers are different depending on whether we address them to students or to teachers. Anyway in both cases history is an artifact that becomes a tool for knowledge if used in an appropriate way. To answer the questions we refer to educational theories. For example, we consider the role of metaphors in constructing concepts and we see that history is a good source of metaphors. Also history helps in connecting different languages and promoting flexibility in students' reasoning

Furinghetti, F. & Paola, D.: 2003, 'History as a crossroads of mathematical culture and educational needs in the classroom', *Mathematics in School*, v.37, n.1, 37-41.

In this paper we present a project for introducing simple elements of probability in secondary school, using the problem of the division of a stake between two players whose game was interrupted before its close. This problem was solved by Luca Pacioli who used proportionality. The problem [together with other problems of chance] was proposed to Pascal and Fermat and from that the theory probability originated. We show that the use of history has to occur in a setting of exploration and classroom discussion in order to enhance students' thinking.

Furinghetti, F. & Radford, L.: 2002 'Historical conceptual developments and the teaching of mathematics: rethinking phylogenesis and ontogenesis', in L. English (editor), *Handbook of international research in mathematics education*, L. Erlbaum Ass., Mahwah, NJ, 631-654.

In this chapter we want to focus on something that the historian Cajori started and in which mathematics educators interested in the history of mathematics are still involved. That is, in considering history not only as a window from where to draw a better knowledge of the nature of mathematics but as a means to transform the teaching itself. The specificity of this pedagogical use of history is that it interweaves our knowledge of past conceptual developments with the design of classroom activities, the goal of which is to enhance students' development of mathematical thinking. We focus on a theme that has been widely discussed over time, that is, the validity of the recapitulation law (ontogeny recapitulates phylogeny), basing our considerations both on theoretical studies and on examples of classroom activities.

Furinghetti, F. & Somaglia, A. M.: 2001, 'The method of analysis as a common thread in the history of algebra: reflections for teaching', *Themes in Education*, v.2, 3-14.

This paper describes the main outline of a course for in-service teachers on the teaching of algebra. In presenting this course our aim was twofold: to make teachers aware of students' difficulties in learning, and to discuss the nature of these difficulties. To summarize, our aim could be defined as "to analyze what algebraic thinking is". To place teachers in a

different perspective, from which to look at didactic problems, we decided to use passages taken from the history of mathematics, without claiming that we were offering a course on the history of algebra. We chose the method of analysis as a common thread. This topic proved to be an efficient way of approaching some critical points of algebraic thinking.

Furinghetti, F. & Somaglia, A. M.: 2003, 'History as a tool for mathematics education and for research in mathematics education', in O.B. Bekken & R. Mosvold (editors), *Study the Masters: The Abel-Fauvel Conference Proceedings* (Kristiansand), 219-233.

In this paper we try to connect the use of history in mathematics education with research in mathematics education. We take as an example the use of history for introducing the tangent as a first step in the approach to derivative. In previous work we have analyzed difficulties in the teaching and learning of these concepts. This analysis revealed specific obstacles, which we take into account in choosing historical passages to be used in the classroom. The reasons for this way of working are explained in the light of possible links with educational theories.

Gregori, S.: 2001, 'Un manoscritto d'abaco in un'esperienza didattica', *L'Educazione Matematica*, a.XXII, s.VI, v.3, 159-172.

In this paper the author intended to describe an experience aimed at introducing history of mathematics in the didactic practice. The direct use of the original text, extracted from an Italian "abacus" manuscript (*Manoscritto Parmense 78* from Biblioteca Palatina, Parma), had important effects on the motivation of students attending the third year of a secondary school (grade 11). Working on the historical sources aroused great interest in the students, who were led in their activity by two different sequences of questions. Those questions were mainly concerned with the comparison of the use of mathematical symbolism, economy and generality of "our" algebra with the methods used by abacists in solving problems involving systems of linear equations. It revealed to be a good opportunity for a "reflection on mathematics", instead of the usual "mathematical reflection".

Grugnetti, L. & Rogers L., with Carvalho e Silva, J., Daniel C., Coray, D., de Guzman, M., Gispert, H., Ismael, A., Jones, L., Menghini, M., Philippou, G., Radford, L., Rottoli, E., Taimina, D., Troy, W. & Vasco, C.: 2000, 'Philosophical, multicultural and interdisciplinary issues', in *ICMI Study*, 39-62.

School mathematics reflects the wider aspect of mathematics as a cultural activity. From a philosophical point of view, mathematics must be seen as a human activity both done within individual cultures and also standing outside any particular one. From an interdisciplinary point of view, students find their understanding both of mathematics and of other subjects enriched through the history of mathematics. From a cultural point of view, mathematical evolution comes from a sum of many contributions growing from different cultures.

Grugnetti, L.: 2000, 'The history of mathematics and its influence on pedagogical problems', in V. Katz (editor), *Using history to teach mathematics: An international perspective*, Mathematical Association of America, 29-35.

In this work historical ideas as they can influence pedagogy in school are discussed. Using the history of mathematics as an introduction to a critical and cultural study of mathematics is one of the most important challenges for both mathematics teachers and students. There are many possibilities in mathematics education for the use of history which are presented here. General examples are given concentrating in particular on the work of Cavalieri and Fibonacci.

Mammana, C. & Tazzioli, R.: 1999, 'The mathematical school in Catania at the beginning of the 20th Century and its influence on didactics', in P. Radelet-De Grave (editor), *Proceedings of third European summer university on "History and Epistemology in mathematical Education"*, Université catholique de Louvain, 223-232.

At the beginning of this century, S. Catania, M. De Franchis, M. Cipolla, G. Marletta and other mathematicians taught at the University of Catania and published textbooks of mathematics at the same time. The mathematician S. Catania wrote textbooks for secondary school, which followed Peano's formalism in mathematics. Cipolla published textbooks on algebra written in a traditional, but rigorous way. Textbooks on geometry were published by De Franchis and Marletta. The content of these textbooks is connected with research on foundations of mathematics and on geometry, which was developed in Italy at that time.

Marchini C.: 2001, 'Evoluzione del pensiero di Francesco Speranza, filosofo della matematica', G. Anichini (editor), *XXI Convegno nazionale UMI-CIIM sull'insegnamento della matematica, NUMI*, v.XXVIII, suppl. n.10, 114-122.

The evolution of Speranza's ideas about Epistemology of Mathematics (which he considered as linked to math education) is outlined. He started with a sort of Bourbakism, but during the years he moved to a revaluation of Enriques, Gonsseth and Bachelard. The last years of Speranza's life were devoted to the analysis of mathematical knowledge and its development as a discipline. This analysis may be defined as epistemological and with it he succeeded in (re)introducing these subjects into the Italian mathematical community, thus (re)establishing a new interest and new forms of co-operation with a more strictly philosophical environment. Epistemology of mathematics was developed by Peano and his school, Enriques, Vailati, Pieri and Beppo Levi. Political and cultural events in Italy caused a long interruption in these studies, only recently re-born. In his last period Speranza took inspiration from Popper and his scholars. Speranza's favorite field was geometry as a sample of an important mathematical theory and a specimen of mathematical empiricism.

Related paper

Marchini C.: 2000, 'The philosophy of mathematics according to Francesco Speranza', in N.A. Malara, P.L. Ferrari., L. Bazzini & G. Chiappini (editors), *Recent Italian research in mathematics education*, Dipartimento di Matematica dell'Università di Modena, 83 -93.

Menghini, M.: 2000, 'On potentialities, limits and risks', in *ICMI Study*, 86-90.

The question of judging the effectiveness of integrating historical resources into mathematics teaching may not be susceptible to the research techniques of quantitative

experimental scientists. It is better handled through qualitative research paradigms such as those developed by anthropologists. Where explicit use of history is concerned, there are limitations and risks. It is seen, through examples, in section 3.7 (by M. Menghini) that it can be difficult to understand the procedure used by a mathematician of ancient times if it is not set within historical context. There is a difficulty here for the teacher to resolve, well before it becomes one for the student. At least two types of danger can arise when using history explicitly. First, using piece-meal historical illustrations can give a false and truncated view of what mathematics, and indeed history, was really like historically. Alternatively, in trying to present a global historical view, we could be in danger of ending up with an education in mathematics history quite independent of the needs of mathematics education.

Paola, D.: 2000, 'Matematica e storia della matematica in classe: la strana storia del teorema degli zeri', *L' Educazione Matematica*.XXI, s.VI, v.2, 144-160.

In this paper we give an example of the use of the history of mathematics in the classroom aimed at giving students the possibility of facing the construction of concepts which are important, but difficult. The concept in question is the approximated calculation of the roots of polynomial equations. An excerpt taken from *Leçons élémentaires sur les mathématiques* by Lagrange (translated into Italian in 1839) is used here.

Pellegrino, C.: 2001, 'Rivisitazioni Geometriche: La prospettiva senza "veli" ovvero Cabri, Monge e la prospettiva', in B. D'Amore (editor), *Didattica della matematica e rinnovamento curricolare*, 219-224

Perspective has been an important link between arts and mathematics which turned out to be quite fruitful. In this paper after a brief overview on the origins of perspective in which we illustrate the *principle of intersection of the visual pyramid* on which it is based, and starting from this simple principle, we give a simple system of perspective representation through the "power" of the basic notions of the *double projection of Monge*, then "get to discover" its fundamental properties and concepts (such as *vanishing points*, *horizon line*, etc.) thanks to the dynamism of *Cabri*. In this way we get to understand the rules that are basis of the various systems of perspective representation as well as the origin of the concepts of *improper point* and *improper straight line* which, beside the *operations of projection* and *section*, are the basis of projective geometry.

Pellegrino, C. & Fiori, C.: 2000a, 'Piani formalmente euclidei', *Rivista Matematica Università di Parma*, n.3, 199-213

In the last decades, university teaching has introduced Geometry as a chapter of Linear Algebra. This implies that the training of mathematicsteachers lacks more and more geometrical culture and spatial intuition, and this fact, of course, has a negative follow-up on the quality of mathematics teaching and its educational value. So it is important to show the links existing between synthetic and analytic approach to the Geometry. In this paper we offer a very general way that, bearing in mind the development of geometrical thought in the last two centuries, allows to shift from one setting to the other. Such procedure, which neither depends on the axioms of order nor on those of congruence (and which

therefore is not necessarily linked to the field of real numbers), allows us to extend the notions of similitude and isometry to a wide class of planes (including the Euclidean one), and to find out and characterize their equations.

Pellegrino, C. & Fiori, C.: 2000b, 'Teoremi ed algoritmi: i due volti del numero', *Progetto Alice*, v.I, 425-439.

This paper, as others of the same authors, deals with cultural aspects of the training of Mathematics teachers by making explicit the relationships, often neglected in specific studies, existing between today' s mathematics and the one of the origins, as well as those between pure and applied mathematics. In particular we consider some aspects of the theory of divisibility in order to highlight the role played by algorithms and problem solving in the long abstraction process that goes from natural numbers to abstract algebra. To make our presentation more impressive and effective, we deal with the heuristic aspects of the topic, use the dialogue form and concentrate in the footnotes all historical and bibliographical references. In particular, to make further research easier, in the references we give more space to as recent and easy-to-find work as possible.

Pellegrino, C. & Brandoli, M.T.: 2001, 'Il principio d'induzione euristicamente parlando', *L'Educazione Matematica*, n.2, 83-93

This paper deals with cultural aspects of the training of mathematics teachers by making explicit the relationships, often neglected in specific studies, existing between today' s mathematics and the one of the origins, as well as those between pure and applied mathematics. In this paper we discuss in a dialogic form one of the aspects that make the application of the principle of induction more problematic, i.e. the formulation of the hypothesis to be demonstrated.

Pergola, M., Zanolì, C., Martinez, A. & Turrini, M.: 2001, 'Modelli fisici per la matematica: sulle sezioni del cilindro retto', *Progetto Alice*, v.II, 143-150.

The aim of this article is to present some mathematical models and mechanical curve-tracing tools related to conic sections. The authors briefly discuss how to use them to organize some didactic activities involving both mathematical aspects and the historical development of mathematics as well.

Pergola, M., Zanolì, C., Martinez, A. & Turrini, M.: 2002a, 'Modelli fisici per la matematica: parallelogrammi, antiparallelogrammi e deltoidi articolati', *Progetto Alice*, v.III, 323-346.

The use of some elementary geometric shapes as parts of machines to construct planar mechanical linkages (physical models or computer simulations), is particularly effective to explain easy planar transformations clearly and generate many curves. Using these instruments, teachers can organize amusing educational activities connected with mathematics history. Moreover these teaching activities produce a profitable connection between empirical experiences and theoretical proofs.

Pergola, M., Zanolì, C., Martínez, A. & Turrini, M.: 2002b, 'Modelli fisici per la matematica: conigrafi flessibili', *Progetto Alice*, v.III, 559-571.

According to contemporary geometricians as conic sections are visible curves that can be drawn on a plane, they are different aspects of the same mathematical object, formed as a result of a long historic process with the contribution of mathematical practitioners. They included the devisers and builders of mechanical instruments suited to draw any kind of conic sections, with simple equipment modifications. Some conic sections drawing devices of this kind are briefly examined in the article.

Pergola, M. & Maschietto, M.: 2003, 'Modelli fisici per la matematica: biellismi del Peaucellier e del Delaunay', *Progetto Alice*, v.IV, 363-382.

In this paper, the mechanisms using parallelograms (and kites) are studied through the presentation of some research studies developed in the second half of the 19th century. The interest (diffused in that period) of mathematicians for articulated mechanisms was connected both to practical problems (development of mechanic engineering) and to theoretical reasons (possibility of using those mechanisms to do geometrical transformations and to study plane curves).

Prosperini, L.: 1999, 'Matematica a teatro: perché no?', *Didattica delle Scienze e Informatica nella Scuola*, a.XXXV, n.199, 28-34.

The author is a mathematics teacher who tries to present to her students some basic facts of mathematics - the role of axioms in a theory, the types of proofs, the role of figures as a stimulus/trap to intuition - through dramatization. She has written prompt-copies to be played in school after having read some famous works (the treatise of history by C. Boyer, the books *Travel through the genius* by W. Dunham, *Great mathematicians* by E. T. Bell).

Radford, L., Boero, P. & Vasco, C.: 2000, 'Epistemological assumptions framing interpretations of students understanding of mathematics', in *ICMI Study*, 162-167.

In using the history of mathematics to better understand student's processes of learning mathematics and the way in which such an understanding can be used in the design of classroom activities, two different phenomena need to be linked. On the one hand, the learning processes of contemporary students (psychological phenomena); on the other hand, the historical construction of mathematical knowledge (historical-epistemological phenomena). The linking of psychological and historical-epistemological phenomena requires a clear epistemological approach. Within the field of mathematics education, different approaches have been used. The aim of this paper is to provide an overview of some approaches and their corresponding epistemological frameworks.

Rottoli, E. & Riva, G.: 2000, 'The antiphairesis: a site of an educational dialogue among Mathematics, History and Philosophy', in W.-S. Horng & F.-L. Lin (editors), *Proceedings of the HPM 2000 conference* (Taipei), v.II, 94-102.

The antiphairesis of two homogeneous quantities is a process that consists in subtracting the smaller of two quantities from the larger one: after each removing, the excess is left in place of the larger quantity while the smaller one stays unchanged. The result is a *logos* (ratio), a couple of rational numbers expressing the relation between the quantities. The use of antiphairesis in didactics helps to reflect on the rise of the concept that later will be formalized in rational number. It offers, at the same time, the opportunity to perceive the wealth of forms of thinking that are associated with this concept and that are linked with manifold concrete situations. Moreover, the practice of antiphairesis in comparing two quantities constitutes the last step of a formation process of sense of quantity that starts since the beginning of elementary school. That process consists of introducing the concept of quantity through concrete activities that allow reasoning about the relations of equality and order only through transformations (without counting, measuring, weighing).

Scimone, A.: 2002, 'Following Goldbach's tracks', in A. Rogerson (editor), *Proceedings of international conference on mathematics education into the 21st Century* (Terrasini), 339-342.

The theoretical framework of this work is the theory of didactic situations in mathematics by Guy Brousseau. In particular, this work concerns an a-didactic situation, namely that part of a didactic situation in which teacher's intention toward pupils is not clear. So, the aim of this research is to analyze some conceptions of pupils while they are facing a conjecture, and in particular a famous historical conjecture like Goldbach's one. So, the historical context is important because it suggests an interplay between history of mathematics and mathematics education. The content of the experimentation grows around validation or falsification of two hypotheses of research: the first one concerning pupils' inability to represent mentally any general method useful for a demonstration; the second one concerning their intuitive ability to recognize the validity of a conjecture. The validation or the falsification of these hypotheses are very useful in order to understand the metacognitive processes which are basic for the learning phase and the cultural growth of pupils.

Related paper

Scimone, A.: 2002, 'La congettura di Goldbach tra storia e sperimentazione didattica', *Quaderno di Ricerca in Didattica*, suppl. n.10, 1-37.

Spagnolo, F.: 1999, 'A theoretical-experimental model for research of epistemological obstacles', in A. Rogerson (editor), *Proceeding of international conference on mathematics education into the 21st Century* (Cairo), v.II, 129-139.

<http://dipmat.math.unipa.it/~grim/21project.htm>

In the theory of situations of G. Brousseau and in general of the French school, the study of epistemological obstacles has had a secondary role. Initially the study of obstacles was tied to questions of an ergonomic nature. How can we distinguish in the shortest possible time the didactic and/or epistemological obstacles relative to the specific content to be taught? Previous models, such as that of Duroux-Brousseau derived from the works of Bachelard, allow us to recognize if a notion is an obstacle. That model however does not provide strong *a-priori* deductive instruments for the analysis of obstacles. In the model of Duroux-Brousseau the research instruments were substantially of a didactic type. Previously, other

research instruments for epistemological obstacles have been distinguished of historical type without any consideration of the obstacles of didactic type.

Spagnolo, F.: 2002, 'Storia delle matematiche, ricerca in didattica ed insegnamento delle matematiche', *La Matematica e la sua Didattica*, n.1, 78-95.

In mathematics education there is an important problem about the relationship between Epistemology, History and communication of mathematics. If interpretation of phenomena of learning/understanding uses a semiotic approach to mathematics, we have the instruments to analyze verbal and non-verbal messages. A classification of semiotic interpretations of the history of mathematics as 1) history of syntax of mathematics languages, 2) history of semantic of mathematics languages, 3) history of pragmatics languages, gives the instruments for an *a priori* interpretation of understanding/learning phenomena. The historic/epistemological representations are the possible routes to pupils' knowledge. In this perspective the history of mathematics is useful to perform research in Mathematics education: researchers in mathematics education may have answers to their problems from historians of mathematics.

Related papers

Spagnolo F.: 2000, 'The role of history of mathematics in research in mathematics education', in A. Rogerson (editor), *Proceeding of international conference on mathematics education into the 21st Century* (Amman), 284-288.

<http://dipmat.math.unipa.it/~grim/21project.htm>

Spagnolo, F.: 2002, 'History and Ethno-Mathematics in the Interpretation of the process of learning/teaching', in *Proceedings of 13° ICMI Study conference*, Faculty of Education, University of Hong Kong, 20-25.

Spagnolo, F.: 2003, 'The role of history in the interpretation of process of learning/teaching', in A. Gagatsis & S. Papastavridis (editors), *Proceedings 3rd Mediterranean conference on mathematics education*, 533-544.

Speranza, F. & Vighi, P.: 2002, 'Alcune idee a proposito di spazio e tempo', in B. Piochi & A. Contardi (editors), *Le difficoltà nell'apprendimento della matematica. Metodologia e pratica di insegnamento*, Centro Erickson, Trento, 135-146

The paper looks at the complex concepts of time and space and the mental tools that human mind uses to deal with them. It looks at the role of space in geometry, the importance of "talking about space" and the representation of space in art. Rather than a definition of space we put forward a series of dichotomies: relational vs. independent, finite vs. infinite, limited vs. unlimited etc.

Testa, G.: 2000, 'L'enseignement des coniques à travers une approche historique: comment saisir un texte?', *Repères IREM*, n.41, 105-119.

The article is about a teaching experience with a group of high school volunteer students who engaged in the discovery of the conics through the reading of an ancient French text by Monsieur de la Chapelle [Rouen, 1710 circa - Paris, 1792]: *Traité des Sections Coniques, et autres courbes anciennes*, Debure, Paris, 1765 [reprinted in: *Reproduction de textes anciens nouvelle série* n.6, IREM VII, Paris, 1994]. In order to overcome the obstacle of reading the text in the original language, a sort of simulated reading was envisaged by means of 34

charts (Parabola, Ellipse and Hyperbola) based on the following guidelines: a) progression of difficulties; b) usage of the original notations of the text, in particular for drawing proportions and parentheses; c) reproduction of the figures in the text; d) literal translation of the text into Italian, trying to preserve the French language structures. The article also provides information on M. de la Chapelle's life, together with teaching guidelines, a few charts with commentary, critical remarks on the results including students' remarks.

Testa, G.: 2001, 'Per un avvio alla ricerca "storica" in campo scientifico: studenti al lavoro', *L'Insegnamento della Matematica e delle Scienze Integrate*, v.24B, 39-68. The article deals with the results of a research into math history and science carried out by a group of students by surfing the net. The results vary according to the different level of the students, some of them being real beginners and very young, others more competent and older. The aims of the experience were: a) to raise students' awareness about surfing and using websites of scientific content suitable for in-depth research; b) to encourage students to research work by reading historical-scientific documents in the original and in a critical way; c) to train students to work independently while researching; d) to offer the students involved and also the students preparing for final exams an example of in-depth project using a multi-media approach, as in the case of the project on Archimedes. The topics of the research are: Archimedes' life and death, contributions from the ancients to the study of optics, Foucault's pendulum.

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